Review Project
MAP 2302
(Spring 2016)

Name: ______________________

Date: ______________________

total: 34 points possible

Solve the equation $y'' - 6y' + 9y = t$ with $y(0) = 0$ and $y'(0) = 4$ using

6 points each

1. Undetermined coefficients
2. Variation of parameters and Wronskians
3. LaPlace transforms

4. Solve the system of equations using LaPlace transforms — 8 points

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + \frac{dy}{dt} = 0$$

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 4\frac{dx}{dt} = 0$$

with $x(0) = 1$, $x'(0) = 0$, $y(0) = -1$, $y'(0) = 5$

5. Solve the linear equation — 4 points

$$xy' + (1+x)y = e^{-x}\cos 2x$$

6. Solve the exact equation — 4 points

$$\left(1 - \frac{3}{y} + x\right)\frac{dy}{dx} + y = \frac{3}{x} - 1$$

UC
1

1) $\boxed{\begin{array}{l} y'' - 6y' + 9y = t \\ y(0) = 0 \\ y'(0) = 1 \end{array}}$

$\boxed{\text{undetermined coefficients}}$

$\boxed{\text{auxilary equation}}$

$m^2 - 6m + 9 = 0 \qquad m_1 = m_2 = 3$

$(m-3)(m-3) = 0$

$$\boxed{y_c = c_1 e^{+3t} + c_2 t e^{+3t}}$$

$\boxed{y_p}$

$y_p = At + B$

$y_p' = A$

$y_p'' = 0$

$0 - 6(A) + 9(At + B) = t$

$-6A + 9At + 9B = t$

$(9A)t + (9B - 6A) = t$

$9A = 1$

$\boxed{A = \frac{1}{9}}$

$9B - 6A = 0$

$9B - 6\left(\frac{1}{9}\right) = 0$

$9B = \frac{6}{9} = \frac{2}{3}$

$\boxed{B = \frac{2}{27}}$

so $\boxed{y_p = \frac{1}{9}t + \frac{2}{27}}$

so $y = y_c + y_p$

$$\boxed{y = c_1 e^{+3t} + c_2 t e^{+3t} + \frac{1}{9}t + \frac{2}{27}}$$

UC
②

$$y = c_1 e^{+3t} + c_2 t e^{+3t} + \frac{1}{9}t + \frac{2}{27}$$

$$y(0) = c_1 e^0 + c_2 (0) e^0 + \frac{1}{9}(0) + \frac{2}{27} = 0$$

$$c_1 + \frac{2}{27} = 0 \qquad \boxed{c_1 = \frac{-2}{27}}$$

$$y'(t) = 3c_1 e^{3t} + c_2 e^{3t} + c_2 \cdot 3e^{3t} \cdot t + \frac{1}{9}$$

$$y'(0) = 3c_1 e^0 + c_2 e^0 + c_2 \cdot 3e^0 (0) + \frac{1}{9} = 1$$

$$3c_1 + c_2 + \frac{1}{9} = 1$$

$$3\left(\frac{-2}{27}\right) + c_2 + \frac{1}{9} = 1 \qquad \boxed{c_2 = \frac{10}{9}}$$

$$\frac{-2}{9} + c_2 + \frac{1}{9} = 1$$

$$c_2 = 1 - \frac{1}{9} + \frac{2}{9} = \frac{10}{9}$$

final answer

$$\boxed{y(t) = \frac{-2}{27} e^{3t} + \frac{10}{9} t e^{3t} + \frac{1}{9}t + \frac{2}{27}}$$

(2) $\boxed{\begin{array}{l} y'' - 6y' + 9y = t \\ y(0) = 0 \\ y'(0) = 1 \end{array}}$ $f(t)$

variation of parameters & wronskians

$\boxed{y_c}$ $m^2 - 6m + 9 = 0$

$(m-3)(m-3) = 0$

$m_1 = m_2 = 3$

$\boxed{y_c = c_1 e^{3t} + c_2 t e^{3t}}$

$\boxed{y_p}$ $y_1 = e^{3t}$ $\quad y_2 = te^{3t}$

$y_1' = 3e^{3t}$ $\quad y_2' = e^{3t} + 3te^{3t}$

$$\boxed{W} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & (e^{3t} + 3te^{3t}) \end{vmatrix}$$

$$= e^{3t}(e^{3t} + 3te^{3t}) - (3e^{3t} \cdot te^{3t})$$

$$= e^{6t} + 3te^{6t} - 3te^{6t} = \boxed{e^{6t}}$$

$$\boxed{W_1} = \begin{vmatrix} 0 & y_2 \\ f(t) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & te^{3t} \\ t & e^{3t} + 3te^{3t} \end{vmatrix}$$

$$= -(te^{3t})(t) = \boxed{-t^2 e^{3t}}$$

$$\boxed{W_2} = \begin{vmatrix} y_1 & 0 \\ y_1' & f(t) \end{vmatrix} = \begin{vmatrix} e^{3t} & 0 \\ 3e^{3t} & t \end{vmatrix}$$

$$= te^{3t} - 0 = \boxed{te^{3t}}$$

VP

2

$$\boxed{u_1'} = \frac{W_1}{W} = \frac{-t^2 e^{3t}}{e^{6t}} = \boxed{-t^2 e^{-3t}}$$

$$\boxed{u_1} = \int -t^2 e^{-3t}\,dt$$

D	I
$+\ t^2$	$-e^{-3t}$
$-\ 2t$	$\frac{1}{3}e^{-3t}$
$+\ 2$	$-\frac{1}{9}e^{-3t}$
$-\ 0$	$\frac{1}{27}e^{-3t}$

$$\boxed{u_1} = \boxed{\frac{1}{3}t^2 e^{-3t} + \frac{2}{9}te^{-3t} + \frac{2}{27}e^{-3t}}$$

$$\boxed{u_2'} = \frac{W_2}{W} = \frac{te^{3t}}{e^{6t}} = te^{-3t}$$

D	I
$+\ t$	e^{-3t}
$-\ 1$	$-\frac{1}{3}e^{-3t}$
$+\ 0$	$\frac{1}{9}e^{-3t}$

$$\boxed{u_2} = \int te^{-3t}\,dt = \boxed{-\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t}}$$

so $y_p = u_1 y_1 + u_2 y_2$

$$= \left(\frac{1}{3}t^2 e^{-3t} + \frac{2}{9}te^{-3t} + \frac{2}{27}e^{-3t}\right)e^{3t} + \left(-\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t}\right)te^{3t}$$

$$y_p = \cancel{\frac{1}{3}t^2} + \frac{2}{9}t + \frac{2}{27} + \cancel{\frac{-1}{3}t^2} - \frac{1}{9}t$$

$$\boxed{y_p = \frac{1}{9}t + \frac{2}{27}}$$

$$\boxed{\begin{array}{c} y = y_c + y_p = c_1 e^{3t} + c_2 te^{3t} + \frac{1}{9}t + \frac{2}{27} \\ \text{with } c_1 = \frac{-2}{27} \quad c_2 = \frac{10}{9} \end{array}}$$

LT
1

LaPlace Transforms

3
$$\boxed{\begin{aligned} &y'' - 6y' + 9y = t \\ &y(0) = 0 \\ &y'(0) = 1 \end{aligned}}$$

$$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{t\}$$

$$s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) - 6[s\mathcal{L}\{y\} - y(0)] + 9\mathcal{L}\{y\} = \mathcal{L}\{t\}$$

$$s^2\mathcal{L}\{y\} - s(\cancel{0}) - 1 - 6s\mathcal{L}\{y\} + 6(\cancel{0}) + 9\mathcal{L}\{y\} = \frac{1}{s^2}$$

0 0

$$(s^2 - 6s + 9)\mathcal{L}\{y\} = \frac{1}{s^2} + 1$$

$$\mathcal{L}\{y\} = \frac{1+s^2}{s^2(s^2-6s+9)} = \frac{1+s^2}{s^2(s-3)^2}$$

$$\mathcal{L}\{y\} = \frac{\cancel{s^2}}{\cancel{s^2}(s-3)^2} + \frac{1}{s^2(s-3)^2} = \frac{1}{(s-3)^2} + \frac{1}{s^2(s-3)^2}$$

$$\boxed{y = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2(s-3)^2}\right\}}$$

$e^{at}f(t) = \mathcal{L}^{-1}(F(s-a))$

part 1

partial fractions

or $\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau$

part 2

or convolutions

part 1

$$\mathcal{L}^{-1}\left\{\left.\frac{1}{s^2}\right|_{s\to s-3}\right\} = \boxed{t \cdot e^{3t}}$$

$n+1=2$

$n=1$

LT

2

part 2

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-3)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\cdot\frac{1}{s^2}\right\}$$

$$= te^{3t} * t$$

$$= \int_0^t \tau e^{3\tau}\cdot(t-\tau)\,d\tau$$

$$= \int_0^t \left(\underbrace{t\tau e^{3\tau}}_{\text{parts}} - \underbrace{\tau^2 e^{3\tau}}_{\text{parts}}\right)d\tau$$

	D	I
+	τ	$e^{3\tau}$
−	1	$\frac{1}{3}e^{3\tau}$
+	0	$\frac{1}{9}e^{3\tau}$

	D	I
+	τ^2	$e^{3\tau}$
−	2τ	$\frac{1}{3}e^{3\tau}$
+	2	$\frac{1}{9}e^{3\tau}$
−	0	$\frac{1}{27}e^{3\tau}$

$$= t\left[\frac{1}{3}\tau e^{3\tau} - \frac{1}{9}e^{3\tau}\right]_0^t - \left[\frac{1}{3}\tau^2 e^{3\tau} - \frac{2}{9}\tau e^{3\tau} + \frac{2}{27}e^{3\tau}\right]_0^t$$

$$= t\left[\left(\frac{1}{3}te^{3t} - \frac{1}{9}e^{3t}\right) - \left(\cancel{\frac{1}{3}(0)e^0}^{\,0} - \frac{1}{9}e^0\right)\right]$$

$$- \left[\left(\frac{1}{3}t^2e^{3t} - \frac{2}{9}te^{3t} + \frac{2}{27}e^{3t}\right) - \left(\underbrace{\frac{1}{3}\cancel{(0)}}_{0} - \underbrace{\frac{2}{9}\cancel{(0)}}_{0} + \frac{2}{27}e^0\right)\right]$$

LT ③

$$= \cancel{\frac{1}{3}t^2e^{3t}} - \frac{1}{9}te^{3t} + \frac{1}{9}t - \cancel{\frac{1}{3}t^2e^{3t}} + \frac{2}{9}te^{3t} - \frac{2}{27}e^{3t} + \frac{2}{27}$$

$$= \boxed{\frac{1}{9}te^{3t} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{2}{27}}$$ part 2

final answer

$$= te^{3t} + \frac{1}{9}te^{3t} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{2}{27}$$

$$= \boxed{\frac{10}{9}e^{3t} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{2}{27}}$$

4 $\boxed{\text{System}}$

$$\boxed{\begin{aligned} x'' + x' + y' &= 0 \\ y'' + y' - 4x' &= 0 \end{aligned}}$$

$x(0) = 1$

$x'(0) = 0$

$y(0) = -1$

$y'(0) = 5$

$$\boxed{\begin{aligned} \mathcal{L}\{x''\} + \mathcal{L}\{x'\} + \mathcal{L}\{y'\} &= \mathcal{L}\{0\} \\ \mathcal{L}\{y''\} + \mathcal{L}\{y'\} - 4\mathcal{L}\{x'\} &= \mathcal{L}\{0\} \end{aligned}}$$

$$\mathcal{L}\{x''\} + \mathcal{L}\{x'\} + \mathcal{L}\{y'\} = 0$$

$$s^2\mathcal{L}\{x\} - s\cdot \cancelto{1}{x(0)} - \cancelto{0}{x'(0)} + s\mathcal{L}\{x\} - \cancelto{1}{x(0)}$$
$$+ s\cdot\mathcal{L}\{y\} - \cancelto{-1}{y(0)} = 0$$

$$s^2\mathcal{L}\{x\} - s + s\mathcal{L}\{x\} - \cancel{1} + s\mathcal{L}\{y\} + \cancel{1} = 0$$

$$\boxed{(s^2+s)\mathcal{L}\{x\} + s\mathcal{L}\{y\} = s}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} - 4\mathcal{L}\{x'\} = 0$$

$$s^2\mathcal{L}\{y\} - s\cdot\cancelto{-1}{y(0)} - \cancelto{5}{y'(0)} + s\mathcal{L}\{y\} - \cancelto{-1}{y(0)}$$
$$- 4\left[s\mathcal{L}\{x\} - \cancelto{1}{x(0)}\right] = 0$$

$$s^2\mathcal{L}\{y\} + s - \cancel{5} + s\mathcal{L}\{y\} + \cancel{1} - 4s\mathcal{L}\{x\} + \cancel{4} = 0$$

$$\boxed{(s^2+s)\mathcal{L}\{y\} - 4s\mathcal{L}\{x\} = -s}$$

system ②

$$\boxed{\begin{aligned}(s^2+s)\,\mathcal{L}\{x\} + s\,\mathcal{L}\{y\} &= s\\ -4s\,\mathcal{L}\{x\} + (s^2+s)\,\mathcal{L}\{y\} &= -s\end{aligned}}$$

Let $A = \mathcal{L}\{x\}$ $\quad B = \mathcal{L}\{y\}$

$$D = \begin{vmatrix} s^2+s & s \\ -4s & s^2+s \end{vmatrix}$$

$$= (s^2+s)(s^2+s) + 4s(s)$$
$$= s^4+2s^3+s^2+4s^2$$
$$= s^4+2s^3+5s^2 = \boxed{s^2(s^2+2s+5)}$$

$$D_A = \begin{vmatrix} s & s \\ -s & s^2+s \end{vmatrix} = s(s^2+s) - (-s^2)$$
$$= s^3+s^2+s^2 = s^3+2s^2$$
$$= \boxed{s^2(s+2)}$$

$$D_B = \begin{vmatrix} s^2+s & s \\ -4s & -s \end{vmatrix} = -s(s^2+s) + 4s(s)$$
$$= -s^3-s^2+4s^2$$
$$= -s^3+3s^2 = \boxed{s^2(3-s)}$$

$$A = \boxed{\mathcal{L}\{x\}} = \frac{D_A}{D} = \frac{s^2(s+2)}{s^2(s^2+2s+5)} = \boxed{\frac{s+2}{s^2+2s+5}}$$

$$B = \boxed{\mathcal{L}\{y\}} = \frac{D_B}{D} = \frac{s^2(3-s)}{s^2(s^2+2s+5)} = \boxed{\frac{3-s}{s^2+2s+5}}$$

system
3

$$\boxed{\mathcal{L}\{x\}} = \frac{s+2}{s^2+2s+5} = \frac{s+2}{(s^2+2s+1)+4} = \frac{s+2}{(s+1)^2+4}$$

$$x = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4} + \frac{1}{(s+1)^2+4}\right\}$$

shift of cos

shift of sin

$$x = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\Big|_{s\to s+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1\cdot 2}{s^2+4}\Big|_{s\to s+1}\right\}$$

$$\boxed{x = e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t}$$

$$\mathcal{L}\{y\} = \frac{-s+3}{s^2+2s+5} = -\left(\frac{s-3}{(s+1)^2+4}\right) = -\left[\frac{s+1}{(s+1)^2+4} - \frac{4}{(s+1)^2+4}\right]$$

$$y = \mathcal{L}^{-1}\left\{\frac{-(s+1)}{(s+1)^2+4}\right\} + 2\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}$$

$$\boxed{y = -e^{-t}\cos 2t + 2e^{-t}\sin 2t}$$

5) $xy' + (1+x)y = e^{-x}\cos 2x$ $\boxed{\text{linear}}$

$$\boxed{y' + \left(\frac{1+x}{x}\right)y = \frac{e^{-x}\cos 2x}{x}}$$

$$\boxed{\mu(x)} = e^{\int P(x)dx} = e^{\int \frac{1+x}{x}dx}$$

$$= e^{\int (\frac{1}{x}+1)dx} = e^{\ln x + x} = e^{\ln x} \cdot e^{x}$$

$$= \boxed{xe^{x}}$$

multiply all parts by $\mu(x)$

$$xe^{x}y' + \left(\frac{1+x}{x}\right)xe^{x}y = \frac{e^{-x}\cos 2x}{x} \cdot xe^{x}$$

$$\frac{d}{dx}\left[\mu(x)\cdot y\right] = \frac{d}{dx}\left[xe^{x}\cdot y\right] = \cos 2x$$

$$xe^{x}\cdot y = \int \cos 2x\, dx$$

$$xe^{x}\cdot y = \frac{1}{2}\sin 2x + c$$

$$\boxed{y = \frac{\sin 2x}{2xe^{x}} + \frac{c}{xe^{x}}}$$

(6) $\left(1-\frac{3}{y}+x\right)\frac{dy}{dx}+y=\frac{3}{x}-1$ $\boxed{\text{Exact}}$

need format $M\,dx+N\,dy=0$

$$\left(1-\frac{3}{y}+x\right)\frac{dy}{dx}=\frac{3}{x}-1-y$$

$$\left(1-\frac{3}{y}+x\right)dy=\left(\frac{3}{x}-1-y\right)dx$$

so $\left(\frac{3}{x}-1-y\right)dx-\left(1-\frac{3}{y}+x\right)dy=0$

$$\left(\frac{3}{x}-1-y\right)dx+\left(\frac{3}{y}-1-x\right)dy=0$$

$\boxed{\text{check for Exact}}$

$$\frac{\partial}{\partial y}\left(\frac{3}{x}-1-y\right)=-1 \qquad \frac{\partial}{\partial x}\left(\frac{3}{y}-1-x\right)=-1$$

$$\int\left(\frac{3}{x}-1-y\right)dx=3\ln|x|-x-xy+g(y)$$

$$\frac{\partial}{\partial y}\left[3\ln|x|-x-xy+g(y)\right]=-x+g'(y)$$

$$-x+g'(y)=\frac{3}{y}-1-x$$

$$g'(y)=\frac{3}{y}-1$$

$$g(y)=\int\left(\frac{3}{y}-1\right)dy=3\ln|y|-y$$

so $3\ln|x| - x - xy + g(y) = C_1$

$$\boxed{3\ln|x| - x - xy + 3\ln|y| - y = C_1}$$

possible rearrangements:

$$3(\ln|x| + \ln|y|) - y - x - xy = C_1$$

$$3\ln|xy| - y - x - xy = C_1$$

$$\ln|xy|^3 - y - x - xy = C_1$$